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Abstract

This paper shows how the fundamental geometric inequality lemma of geometric programming can be obtained immediately from information theoretic methods. This results in a drastic simplification of the proof and points the way to other connections between information theory and geometric programming

Key Words

Geometric programming, information theory, geometric inequalities, duality.

Introduction

This note shows how information theoretic methods can be used to obtain simply the fundamental geometric inequality lemma. This lemma is the base used by Duffin, Peterson and Zener (1967) for proving their duality results for geometric programming, and its proof occupies three pages in their book. Their method of proof is a technically impressive procedure. However to illustrate the power of information theoretic methods we shall give a simple proof of this fundamental inequality. The approach we shall take is to show the connection between geometric programming and constrained Khinchine-Kullback-Leibler, or minimum discrimination information (MDI) estimation. Complete duality states for MDI estimation are known (c.f. Brockett, Charnes and Cooper (1980), Charnes, Cooper and Seiford (1978), Charnes and Cooper (1974a,b)). Additionally, in the usual duality state of interest, the statistical properties of the solution are known, facilitating sensitivity analysis. The computation of MDI estimation is easily performed using an unconstrained dual convex program involving only linear and exponential functions.

The information theoretic approach is based upon the mean information for discriminating between two densities f_1 and f_2 (relative to some fixed dominating measure λ). The mean information for discrimination in favor of f_1 against f_2 is defined by (c.f. Kullback (1959)).

$$I(f_1 | f_2) = \int_0^\infty f_1(x) \ln \left[\frac{f_1(x)}{f_2(x)} \right] \lambda(dx).$$

Applying Jensen's inequality to the convex function $h(y) = y\ln y$ with the random variable $Y = \frac{f_1(X)}{f_2(X)}$, and assuming that X has probability measure $f_2(x)\ \lambda(dx)$ yields the result

$I(f_1|f_2) \geq 0$ with $I(f_1|f_2) = 0$ if and only if $f_1 = f_2$ a.s. $[\lambda]$. Amazingly, this is the only result needed to give a proof to the so-called geometric inequality (c.f. Duffin, Peterson and Zener (1967) pg 110). As usual, 0ln0 is taken to be 0.

Lemma 1 (Geometric Inequality-Duffin, Peterson and Zener (1967))

Suppose $\underset{\sim}{x} \in \mathcal{R}^n$, and $\underset{\sim}{\delta} \in \mathcal{R}^n$ with $\delta_i \geq 0$, $i = 1,2, \ldots, n$.

Then

$$\Sigma x_i \delta_i + (\Sigma \delta_i) \ln(\Sigma \delta_i) \leq (\Sigma \delta_i) \ln(\Sigma \exp\{x_i\}) + \Sigma \delta_i \ln \delta_i$$

with equality if and only if there exists a non-negative number B such that

$$\delta_i = B\exp\{x_i\},\ i = 1,2,\ldots,n.$$

Proof: Consider two probability distributions P and Q over $\{1,2,\ldots,n\}$ given by

$$p_i = \frac{\delta_i}{\Sigma \delta_i} \text{ and } q_i = \frac{\exp\{x_i\}}{\Sigma \exp\{x_i\}}.$$

Then since $I(P|Q) \geq 0$, and consequently

$$0 \leq (\Sigma \delta_i) I(P|Q) = \Sigma \delta_i \ln \left\{ \frac{\delta_i (\Sigma \exp\{x_i\})}{(\Sigma \delta_i)(\Sigma \exp\{x_i\})} \right\} = (\Sigma \delta_i) \ln (\Sigma \exp\{x_i\}) + \Sigma \delta_i \ln \delta_i -$$

$(\Sigma \delta_i) \ln (\Sigma \delta_i) - \Sigma \delta_i x_i$ with equality if and only if $p_i = q_i$, i.e., $\delta_i = B\exp\{x_i\}$, $i = 1, 2, \ldots, n$.

Q.E.D.

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